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Comparison of Evaluation Procedures for the Subcritical Crack Growth Parameter *n*

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Abstract

The uncertainty in the determination of the exponent n *in the equation of the subcritical crack velocity from four-point-bending experiments is investigated with respect to different evaluation methods. If the bending strength values are Weibull distributed, and the crack-extension parameter* n *is calculated by linear regression of the bending strength values of a number of experiments at different loading rates, an analytical solution can be given for the mean value and the standard deviation. It turns out that both the mean value and the standard deviation depend on the Weibull modulus* m *and the true value* n_0. *The analytical solution illustrates the essential features of this dependence on* m *and* n_0. *For other evaluation methods, e.g. the one proposed as the CEN standard, this dependence is investigated by a Monte-Carlo simulation for different crack-extension parameters* n_0 *and different Weibull moduli* m. *The standard deviation, which is calculated, is the theoretical lowest limit for certain evaluation procedures. By this, the estimation of the margin of error is put on a firm ground. Since the standard deviation increases with* n_0, *there is only a limited range in which* n *can be determined by four point-bending tests. A new evaluation method, which gives a better approximation than the method proposed as the CEN standard, is presented. The computational effort of this evaluation method is only slightly larger. It furthermore allows the number of experiments to be analytically calculated, which is necessary to obtain a certain accuracy.*

Die Unsicherheit in der Bestimmung des Exponenten n *in der Gleichung der unterkritischen Rißausbreitung aus Vier-Punkt-Biegeversuchen wird bei verschiedenen Auswertemethoden untersucht. Falls die Biegefestigkeitswerte Weibull-verteilt sind und der Rißausbreitungsparameter* n *durch lineare Regression der Biegefestigkeitswerte einer Anzahl von Versuchen bei verschiedenen Lastraten bestimmt wird, kann eine analytische Lösung für den Mittelwert und die Standardabweichung angegeben werden. Es stellt sich heraus, däß sowohl der Mittelwert als auch die Standardabweichung vom Weibull-Modul* m *und vom ursprünglichen Wert* n_0 *abhängen. Die analytische Lösung verdeutlicht die wesentlichen Zusammenhänge. Für andere Auswertemethoden, wie z. B. die als CEN-Standard vorgeschlagene, wurde diese Abhängigkeit für verschiedene Rißausbreitungsparameter* n_0 *und verschiedene Weibull-Moduli* m *mit einer Monte-Carlo-Simulation untersucht. Die berechnete Standardabweichung ist die theoretische untere Grenze für ein bestimmtes Auswerteverfahren. Damit wird die Fehlerabschätzung auf eine solide Basis gestellt. Da die Standardabweichung mit* n_0 *ansteigt, gibt es nur einen eingeschränkten Bereich, in dem* n*-Werte aus Vier-Punkt-Biegeversuchen bestimmt werden können. Eine Auswertemethode wird vorgestellt, die eine bessere Näherung ist, als die für den CEN-Standard vorgeschlagene, und nur geringfügig höheren Rechenaufwand erfordert. Diese erlaubt weiters eine analytische Berechnung der Anzahl von Versuchen, die notwendig ist, um eine bestimmte vorgegebene Genauigkeit zu erreichen.*

On étudi l'incertitude sur la détermination de l'exposant n *de l'équation de vitesse de propagation des fissures sous-critiques utilisée dans les tests en quatre points ceci en fonction des méthodes d'évaluation employées. Si la résistance à la flexion suit une loi de distribution de Weibull, et que le paramètre* n *d'extension de la fissure est calculé par régression linéaire à partir de valeurs de résistance à la flexion issues de plusieurs expériences effectuées à des vitesses de chargement différentes, il est possible de donner une expression analytique de la valeur moyenne et de l'écart-type. En fait, aussi bien la valeur moyenne que l'écart-type dépendent du module de Weibull* m *et de* n_0. *La solution analytique donne les caractéristiques essentielles de cette*

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dépendance en m *et* n_0. *Pour d'autres méthodes d'évaluation, i.e. celle proposée comme standard CEN, on étudie cette dépendance par simulation de type Monte-Carlo pour différents paramètres d'extension de la fissure* n_0, *et différents modules de Weibull* m. *L'écart-type que nous calculons est la limite inférieure théorique pour certaines procédures d'évaluation. On peut ainsi fonder l'estimation de l'erreur sur une base sûre. L'écart-type augmentant avec* n_0, n *ne peut être déterminé par des tests en quatre-points que dans une certaine fourchette. On présente une nouvelle méthode d'évaluation qui fournit une approximation meilleure que celle du standard CEN. Cette méthode d'évaluation ne réclame qu'un léger supplément de calcul informatique. De plus, il permet de calculer analytiquement le nombre d'expériences nécessaires pour obtenir une précision donnée.*

1 Introduction

The strength of ceramics mainly depends on the length of cracks and pores, which are present in the volume or on the surface of the material. In most ceramics, these cracks extend at room temperature in a corrosive atmosphere (e.g. water vapour). If a is the crack length and K_I the stress-intensity factor in fracture mode I, the dependence of the velocity of the crack extension $\dot{a}$ is usually described by the power law

$$\dot{a}(t) = AK_I(t)^{n_0} \tag{1}$$

where the dependence on the K factor is due to the breaking of stretched bonds at the crack tip.[1] It has been observed that the power law is able to describe the crack extension in ceramics in a wide range of loading rates and in different environments.[2] Because the lifetime of a ceramic is limited by these two parameters A and n_0, their determination is of particular interest. A common way to determine the crack-extension parameters is to perform bending tests at different loading rates.

In recent years the European Committee for Standardization (CEN) and the German Institute for Standardization (DIN) have made great efforts to develop standardized testing methods for ceramics, e.g. the testing of bending strength[3] and fracture toughness.[4] A proposal for the determination of the crack-extension parameters by four-point-bending tests has been worked out.[5]

In this work the dependence of the crack-extension parameter n, which denotes the crack-extension parameter obtained by a number of Weibull-distributed bending strength values, on the Weibull modulus m and on the true value n_0 is analytically given for the case of a linear regression of the measured values at different loading rates. From these equations the main features can be seen. For the evaluation proposed as the CEN standard (as well as for other possibilities of evaluation), the dependence is tested by a Monte-Carlo simulation, which, if no analytical solution can be found, is commonly used to compare different evaluation methods (e.g. the bending strength of ceramics by Steen *et al.*[6]).

It is shown that four-point-bending tests have only a limited applicability to determine the crack-extension parameter n. Only taking into account that the measured values are Weibull-distributed at each loading rate, the margin of error increases with increasing n and decreases with increasing m. In this work the limit is set to be 20%. This is confirmed by the CEN standard draft,[5] where a typical error of the linear regression procedure of 20% is seen as an acceptable limit. A standard deviation of more than 20% is seen as impracticable, because such ill-defined values lead to extremely different results in lifetime calculations.[7] In particular, n values above 100, which are obtained for materials such as RSiC,[8] are questionable, if only fifty specimens (which is prescribed by the CEN standard) are tested. The CEN standard draft[5] states that the procedure should only be used for n values lower than 80. The calculations in this paper, however, show that the limit is dependent on the Weibull modulus m and may be much less (e.g. 30 for $m = 10$, see Section 4).

A better mathematical evaluation procedure is presented in Section 5. On the one hand, this procedure allows the range to be extended, in which n can be determined by four-point-bending tests with respect to the limit already stated; on the other hand this evaluation procedure allows the number of experiments, which is required to obtain a certain given accuracy, to be calculated in advance.

2 Theoretical Background

The following two assumptions have to be made: firstly, the crack velocity obeys a power law, see eqn (1), with constant n_0. If n_0 varies due to different failure mechanisms, such as subcritical crack extension due to a viscoelastic behaviour of the second phase[9] or pore growth,[10] only that regime in which n_0 is constant should be regarded for further evaluation.

If the dependence of the K factor on the applied time-dependent stress $\sigma(t)$ is defined by

$$K_I(t) = \sigma(t)Ya^{1/2}(t) \tag{2}$$

with Y being a geometry-dependent factor and a

the crack length, the second assumption is that a critical stress-intensity factor exists,

$$K_{IC} = s(t)Ya^{1/2}(t) \tag{3}$$

where s is the (time-dependent) defect strength. At the time of fracture t_f, $\sigma(t_f)$ and $s(t_f)$ coincide, $\sigma(t_f) = s(t_f)$, or likewise $K_I(t) = K_{IC}$. For a power law dependence of the subcritical crack extension, one obtains[11]

$$s(t) = \left(s^{n_0-2}(0) - \frac{1}{B}\int_0^t dt' \, \sigma^{n_0}(t')\right)^{1/(n_0-2)} \tag{4}$$

B is related to the factor A from eqn (1) by

$$B = \frac{2}{AY^2(n_0 - 2)} K_{IC}^{2-n_0} \tag{5}$$

The defect strength at time $t = 0$, $s(0)$, is commonly denoted as the inert strength. Equation (4) can be solved analytically for a constant stress rate $\sigma(t) = \dot{\sigma}t$. Thus the well-known relation for the bending strength at the fracture time t_f results:

$$\sigma^{n_0+1}(t_f) = (n_0 + 1)\dot{\sigma}B(s^{n_0-2}(0) - s^{n_0-2}(t_f)) \tag{6}$$

The second term on the right hand side is usually omitted. This is valid only if subcritical crack growth leads to a decrease of the defect strength at time t_f. In this case, the term representing the inert strength dominates by far the one of the defect strength at the time t_f, because at room temperature n_0 is very high in most of the ceramics. Hence eqn (6) is simplified:

$$\log \sigma(t_f) = \frac{1}{n_0 + 1} \log \dot{\sigma} + \text{const.} \tag{7}$$

From eqn (7) the crack extension parameter n_0 is obtained in a simple way by plotting the logarithm of the bending strength values versus the logarithm of the loading rates. In such a diagram n_0 and the slope k_0 of a regression line of the bending strength values are related by

$$n_0 = \frac{1}{k_0} - 1 \tag{8}$$

The following discussion is restricted to the case where measured data for the crack extension parameter obey the simplified equations already stated. This means that all measured data decrease with decreasing loading rate and keep off the plateau region of the inert strength.

It was now proposed for the CEN standard that tests have to be performed at five loading rates, each differing by one order of magnitude. For each of these loading rates ten tests are required. The mean value of these experiments is used for a linear regression fit. The crack-extension parameter n is then given analogously to eqn (8),

$$n = \frac{1}{k} - 1 \tag{9}$$

with k being the fitted slope of the regression line.

From now on n and k are the crack-extension parameters obtained by measurements and the respective evaluation procedure, while n_0 and k_0 are the true values.

From this point of view, it is by no means clear that n and n_0 are identical. If the measured values are Weibull-distributed at each loading rate, the margin of error of n increases with decreasing Weibull modulus m. Furthermore, the slope k of higher n values is closer to zero, see eqn (9). From this non-linear relationship a statistical deviation in the measurement of k, which is inevitable because of the inherent properties of ceramics due to statistical failure, leads to a higher uncertainty for high n than for small n. Therefore the experimentally determined crack-extension parameter n is dependent on both the Weibull modulus m and the true value n_0!

It is now supposed that the failure probability of the inert strength obeys a two-parametric Weibull law with scale parameter σ_{in} and Weibull modulus m:

$$P_f = 1 - \exp\left[-(s(0)/\sigma_{in})^m\right] \tag{10}$$

The mean inert strength $\bar{\sigma}_{in}$ is related to the scale parameter σ_{in} of the inert strength by

$$\bar{\sigma}_{in} = \sigma_{in}\Gamma\left(1 + \frac{1}{m}\right) \tag{11}$$

Usually tests are carried out at a high loading rate to provide from crack extension. If crack extension occurs, another Weibull distribution with a loading rate dependent scale parameter σ_β and another Weibull modulus m_* develops from the original distribution:

$$P_f = 1 - \exp\left[-(\sigma(t_f)/\sigma_\beta)^{m_*}\right] \tag{12}$$

Now eqn (6) is rewritten for a certain loading rate $\dot{\sigma}_\beta$:

$$\sigma^{n_0+1}(t_f) = (n_0 + 1)\dot{\sigma}_\beta B s^{n_0-2}(0) \tag{13}$$

Inserting eqn (13) into eqn (10) and comparing with eqn (12), the parameters m_* and σ_β thus are related to the true Weibull modulus m and the inert scale parameter σ_{in} by

$$\sigma_\beta = \sigma_{in}^{(n_0-2)/(n_0+1)}((n_0 + 1)B\dot{\sigma}_\beta)^{1/(n_0+1)} \tag{14}$$

$$m_* = m\frac{n_0 + 1}{n_0 - 2}$$

From this equation the loading rate dependence of the scale parameter σ_β follows:

$$\log \sigma_\beta = \frac{1}{n_0 + 1} \log \dot{\sigma}_\beta + \text{const.} \tag{15}$$

If the Weibull modulus m is known, the loading rate dependent probability function is now given by eqn (12), with m_* related to m by eqn (14).

3 Linear Regression of All Measured Values

A linear regression of all measured values is now investigated, because an analytical solution can be found for this case. From this solution one can understand the essential features, i.e. the dependence of n on m and n_0.

Now experiments are carried out at N loading rates, each differing by one order of magnitude. They are described by the parameter β running from 0 to $N-1$. Without loss of generality, the highest loading rate $\dot{\sigma}_\beta$ can be normalized to one as well as the highest scale parameter σ_β. It can be shown that the slope of a linear regression is unchanged by this procedure. The reason being is that in general a multiplication with an arbitrary value leads only to a shift in a diagram of logarithmic stresses versus logarithmic loading rates. Then the scale parameters and the loading rates have the simple form:

$$x_\beta = \log \dot{\sigma}_\beta = -\beta \quad \beta = 0 \ldots N-1$$

$$y_\beta = \log \sigma_\beta = -\frac{\beta}{n_0+1} \qquad (16)$$

Therefore, in the following sections, eqn (16) is used for the definition of the loading rates and the scale parameter. The slope of a regression line $\mathcal{K}$ is obtained by collecting $x \equiv (x_\beta)$, etc., with $\langle z \rangle = 1/N \sum_{\beta=0}^{N-1} z_\beta$:

$$\mathcal{K} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2} \qquad (17)$$

Inserting non-normalized values into this equation leads to the same slope as normalized values.

3.1 Expectation value and standard deviation of the slope *k*

Through experimental measurements one obtains a set of strength values $\sigma_{\beta j}$ for N different loading rates, being labelled by β, each of the sets consisting of M values being a distribution dependent on n_0 and m_*. Hence, β runs from 0 to $N-1$, j from 1 to M. It is useful to define parameters $\rho_{\beta j}$, which are related to the strength values and the scale parameter by

$$\rho_{\beta j} = \frac{\sigma_{\beta j}}{\sigma_\beta} \qquad (18)$$

Then the dependence on n_0 and k_0 respectively is explicit. By eqn (17) the slope of a linear regression of all values can be calculated:

$$\mathcal{K}(\rho_{01}, \ldots, \rho_{N-1,M}) \equiv \mathcal{K}(\rho) = k_0 + \sum_{\beta=0}^{N-1} \sum_{j=1}^{M} c_\beta \log \rho_{\beta j} \qquad (19)$$

Here the short-hand notation $(\rho_{\beta j}) \equiv \rho$ was introduced. The coefficients c_β are given by:

$$c_\beta = \frac{6(N-1-2\beta)}{MN(N^2-1)} \qquad (20)$$

An immediate consequence of this definition is

$$\sum_{\beta=0}^{N-1} c_\beta = 0 \qquad (21)$$

Equation (19) is rewritten as:

$$\mathcal{K}(\rho) = k_0 + u(\rho) \qquad (22)$$

The variables $\rho_{\beta j}$ obey individual Weibull distributions eqn (12). Hence expectation values of measurements are predicted as integrations with measure

$$\mathrm{d}W = \prod_{\beta,j} \mathrm{d}\rho_{\beta j} P'(\rho_{\beta j}) \qquad (23)$$

If k denotes the outcome of a single measurement of $\mathcal{K}(\rho)$, the corresponding distribution function is given by

$$f(k) = \underbrace{\int \prod_{\beta,j} \mathrm{d}\rho_{\beta j} P'(\rho_{\beta j})}_{\mathrm{d}W} \delta(\mathcal{K}(\rho) - k) \qquad (24)$$

It can be shown that the expectation value of $\mathcal{K}(\rho)$ is the true value k_0. The expectation value of k is then obtained by integration:

$$\begin{aligned} \langle k \rangle &= \int \mathrm{d}k\, k f(k) = \int \mathrm{d}W\, k \int \mathrm{d}k\, \delta(\mathcal{K}(\rho) - k) \\ &= \int \mathrm{d}W \mathcal{K}(\rho) = k_0 + \langle u \rangle \\ &= k_0 + \sum_j \sum_\beta \frac{c_\beta}{\ln 10} \int \mathrm{d}W \ln \rho_{\beta j} \end{aligned} \qquad (25)$$

The factor ln 10 arises because of the change from decadic to natural logarithms. By this, the integral can be solved analytically:

$$\int \mathrm{d}W \ln \rho_{\beta j} = \int_0^\infty \mathrm{d}\rho \underbrace{m_* \rho^{m_*-1} e^{-\rho^{m_*}}}_{P'(\rho)} \ln \rho$$

$$= \frac{1}{m_*} \int_0^\infty \mathrm{d}v\, e^{-v} \ln v = -\frac{\gamma}{m_*} \qquad (26)$$

for all β and j. The fact that the integral over all variables $\rho_{\alpha i} \neq \rho_{\beta j}$ gives unity was used here. The number in the term on the right-hand side is Euler's gamma, $\gamma \sim 0{\cdot}577\,216$. Thus the expectation value of k turns out to be:

$$\langle k \rangle = k_0 - \frac{\gamma}{m_*(\ln 10)} \sum_j \underbrace{\sum_\beta c_\beta}_{0} = k_0 \qquad (27)$$

as already stated. Therefore the expectation value of the slope of the linear regression gives the true value k_0. This means that the measured values of the slopes are distributed around the true value k_0, which is a satisfactory result for the evaluation procedure.

Let us now proceed with the second important parameter, the standard deviation. It is a measure for the accuracy of the evaluation procedure and obtained by taking the square root of the variance $(\Delta k)^2$:

$$(\Delta k)^2 = \langle k^2\rangle - \langle k\rangle^2 = \langle k_0^2 + 2k_0 u + u^2 - k_0^2\rangle$$
$$= 2k_0\underbrace{\langle u\rangle}_{0} + \langle u^2\rangle = \langle u^2\rangle \quad (28)$$

By this, the standard deviation of k is simply the square root of $\langle u^2\rangle$. The procedure to compute $\langle u^2\rangle$ is closely related to eqns (24) and (25):

$$\langle u\rangle^2 = \frac{1}{(\ln 10)^2}\sum_{\beta,j}\sum_{\alpha,i}\int \mathrm{d}W \ln \rho_{\beta j} \ln \rho_{\alpha i} \quad (29)$$

In the case $\beta_j = \alpha i$ the integral can be expressed in terms of the second derivative of the Gamma function $\Gamma(z)$ with respect to z at $z = 1$, otherwise the integral is given by the square of eqn (26). Thus it follows

$$\int \mathrm{d}W \ln \rho_{\beta j} \ln \rho_{\alpha i} = \frac{1}{m_*^2}\left(\gamma^2 + \delta_{\beta j}\delta_{\alpha i}\frac{\pi^2}{6}\right) \quad (30)$$

δ being the Kronecker symbol. Inserting eqn (30) into eqn (29), it is found for the variance of k that

$$(\Delta k)^2 = \frac{\pi^2}{6}\frac{1}{m_*^2(\ln 10)^2}\sum_{\beta,j}(c_\beta)^2$$
$$= 2\pi^2\frac{1}{m_*^2(\ln 10)^2}\frac{1}{MN(N^2-1)} \quad (31)$$

From this equation the standard deviation Δk can be calculated for an arbitrary number of tests M at an arbitrary number of loading rates N, the dependence on M and N being explicit. This is the theoretical lowest limit for the respective evaluation procedure and only a consequence of the Weibull-distributed failure probability of ceramics. The standard deviation of k therefore is inversely proportional to m_*, which is related to the true Weibull modulus m measured at a high loading rate by eqn (14). It should be noted that this relation is valid for all other evaluation procedures presented in the following sections. To reduce the standard deviation, one has to increase the number of tests and/or the number of loading rates.

3.2 Expectation value and standard deviation of the crack extension parameter *n*

Unfortunately, the relation for the evaluation of $\langle n\rangle$ is more difficult. The expectation value is obtained, if the same procedure as presented in Section 3.1 is applied, by

$$\langle (n+1)^2\rangle = \left\langle\frac{1}{(k_0+u(\rho))^2}\right\rangle = \int \mathrm{d}W\frac{1}{(k_0+u(\rho))^2}$$
$$= \frac{1}{k_0^2}\int \mathrm{d}W\left(1 + 3\left(\frac{u}{k_0}\right)^2 + 5\left(\frac{u}{k_0}\right)^4 + \cdots\right) \quad (32)$$

The integrand was expanded into a series, which is only valid for $|u| < k_0$. Denoting $\langle (u/k_0)^l\rangle = f_l$, the standard deviation is obtained using eqn (9):

$$\langle (n+1)^2\rangle = \left\langle\frac{1}{(k_0+u(\rho))^2}\right\rangle$$
$$= (n_0+1)^2(1 + 3f_2 + 5f_4 + \cdots)$$
$$\langle n+1\rangle^2 = \left\langle\frac{1}{k_0+u(\rho)}\right\rangle^2$$
$$= (n_0+1)^2(1 + 2f_2 + f_2^2 + \cdots)$$
$$(\Delta n)^2 = \langle (n+1)^2\rangle - \langle n+1\rangle^2$$
$$= (n_0+1)^2(f_2 + 3f_4 - f_2^2 + 5f_6 - 2f_2 f_4 + - \cdots) \quad (33)$$

From the last equation the variance and thus the standard deviation can be obtained up to sufficiently high accuracy. In practice, it is very tedious but straightforward to calculate the higher orders. Hence, f_l was computed only up to $l = 6$. This approximation was tested by a Monte-Carlo simulation described in Section 4. The accuracy $((\Delta n(\text{approx.}) - \Delta n(\text{Monte-Carlo}))/\Delta n(\text{Monte-Carlo}))$ was proven to be better than 2·5% in the regime where the standard deviation does not exceed 20% of n ($\Delta n/n < 0{\cdot}2$). If one takes only the first term f_2 into consideration, which is obtained by a Gaussian propagation of errors, the accuracy of the approximation is below 20% in the worst case in the regime already described.

To calculate the higher orders of the approximation, an appropriate index symmetrization was imposed. The integral for the fourth order is then

$$\int \mathrm{d}W \ln \rho_{\beta j} \ln \rho_{\alpha i} \ln \rho_{\delta k} \ln \rho_{\epsilon l}$$
$$= \gamma^4 + \gamma^2\gamma_2(\delta_{ij} + \delta_{ik} + \delta_{il} + \delta_{jl} + \delta_{kl})$$
$$+ \gamma_2^2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$- (\gamma\gamma_3 + 3\gamma^2\gamma_2)(\delta_{ijk} + \delta_{ijl} + \delta_{ikl} + \delta_{jkl})$$
$$+ (\gamma_4 - 3\gamma^2 + 4\gamma\gamma_3 + 6\gamma^2\gamma_2)\delta_{ijkl} \quad (34)$$

Here all combinatoric possibilities have to be taken into consideration. The symbol δ_{ijkl} (and δ_{ijk} likewise) introduced in eqn (34) has to be understood as being $\delta_{ijkl} = 1$ for $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. The coefficients γ_l are obtained as derivations of the gamma function:

$$\gamma_l = \left.\frac{\mathrm{d}^l\Gamma(z)}{\mathrm{d}z^l}\right|_{z=1} - (-1)^l\gamma^l \quad (35)$$

Taking eqn (21) into account, only two terms of the integral differ from zero. Therefore the term of the fourth order is thus reduced to

$$f_4 = \left\langle\left(\frac{u}{k_0}\right)^4\right\rangle = \left(\frac{n_0+1}{m_*(\ln 10)}\right)^4$$
$$\times\sum_\beta (c_\beta^4)(\gamma_4 - 3\gamma^2 + 4\gamma\gamma_3 + 6\gamma^2\gamma_2) + 3c_\beta^2\gamma_2^2 \quad (36)$$

The number of the combinatoric possibilities increases very fast with l and the full solution of the integral is quite complex. Hence, only the result for the sixth order is given:

$$f_6 = \left(\frac{n_0 + 1}{m_*(\ln 10)}\right)^6$$
$$\times\sum_\beta (a(c_\beta^6) + 15b(c_\beta^4 c_\beta^2) + 20f(c_\beta^3)^2 + 90g(c_\beta^2)^3) \quad (37)$$

The coefficients were determined by the solution of an appropriate system of equations:

$$a = 15\gamma^4\gamma_2 + 20\gamma^3\gamma_3 - 180\gamma^2\gamma_2^2 + 15\gamma^2\gamma_4 - 120\gamma\gamma_2\gamma_3 + 30\gamma_2^3 - 15\gamma_2\gamma_4 - 10\gamma_3^2 + 6\gamma\gamma_5 + \gamma_6$$
$$b = \gamma_2(6\gamma^2\gamma_2 + 4\gamma\gamma_3 - 3\gamma_2^2 + \gamma_4)$$
$$f = \tfrac{1}{2}(9\gamma^2\gamma_2^2 + 6\gamma\gamma_2\gamma_3 + \gamma_3^2)$$
$$g = \frac{\gamma_2^3}{6} \quad (38)$$

In practice, only the Weibull modulus m of the inert strength is known, which is related to m_* by eqn (14). Hence, in all the following tables the dependence on the Weibull modulus m of the inert strength and not on m_* is given. This is done for practical reasons, because the inert modulus m of a material is usually known. Thus, one can directly see the minimal standard deviation that can be obtained for a certain evaluation procedure.

As an example, Table 1a shows the standard deviation of n for the case of ten experiments at five different loading rates, which is the proposed number of tests for the CEN standard, calculated by the Monte Carlo simulation. If the standard deviation Δn exceeds 20% of n, it is omitted from Table 1a, because it is seen as impracticable to use such ill-defined values for lifetime calculations.

If a number of experiments performed with the same material by different laboratories is available, a mean value can be calculated. For this calculation it is important to use the slopes k of the linear regressions and not the n values. As can be seen by eqn (27), the true value results from the procedure

$$n_0 + 1 = \frac{1}{\langle k \rangle} \equiv \frac{1}{k_0} \quad (39)$$

However, if the mean value is computed by

$$\langle n + 1 \rangle = \left\langle \frac{1}{k} \right\rangle \neq \frac{1}{\langle k \rangle} \quad (40)$$

this leads to a quite different result:

$$\langle n + 1 \rangle = \left\langle \frac{1}{k_0 + u(\rho)} \right\rangle = (n_0 + 1)\int dW\left\langle 1 - \frac{u}{k_0} + \left(\frac{u}{k_0}\right)^2 + \cdots \right\rangle = (n_0 + 1)(1 + f_2 + f_4 + \cdots) \quad (41)$$

The effect of the higher orders of f_i in eqn (41) is small for low n and high m. Otherwise, there can be a considerable difference by calculating the mean value of different series of measurements according to eqn (39) or eqn (40). In Table 1b this is shown for the case of ten experiments at each of the five loading rates. The n values, which one has to expect, exceed the true values n_0 by up to 3·5% for a standard deviation below 20% ($\Delta n/n < 0{\cdot}2$). Thus it is obvious that eqn (39) is much more appropriate to obtain the true value of n_0.

Table 1(a). Standard deviations for a simulation of n-values calculated by a linear regression of 10 values at each of the 5 loading rates

m/n	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*
10	0·494	2·20	5·44							
15	0·328	1·43	3·38	6·34	10·8					
20	0·245	1·06	2·48	4·56	7·38	11·2				
25	0·196	0·847	1·97	3·58	5·74	8·51	12·0	16·4		

3.3 Concluding remarks

The main result of this section is that $\langle k \rangle$ gives the true value of k_0 (and so do all other evaluation procedures presented later). The crucial point for any evaluation procedure is the accuracy of the standard deviation, which is inversely dependent on m_* and thus on the Weibull modulus m. The standard deviation Δn is in a first approximation proportional to $(n_0 + 1)^2/m_*$, where n_0 is the true value of the crack-extension parameter. To get a sufficiently precise result for the standard deviation, the higher orders have to be included, see eqn (33). If for one material more n values of different sets of experiments are available, one has to compute the mean value of n from the slopes of the linear regressions, as proposed in eqn (39), to get the correct result.

4 Linear Regression of the Mean Values at Each loading rate

Now another possibility to evaluate the bending strength values to compute the crack-extension parameter n is investigated. The evaluation procedure proposed for the CEN standard is to test ten specimens at five loading rates. A linear regression of the mean values at each loading rate is performed, thus resulting in a slope k. From

Table 1(b). Mean value according to eqn (40) for a simulation of n-values calculated by linear regression of 10 values at each of the 5 loading rates

m/n	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*
10	10·02	20·2	30·8							
15	10·01	20·1	30·4	40·9	51·8					
20	10·00	20·1	30·2	40·5	51·0	61·7				
25	10·00	20·0	30·1	40·3	50·6	61·1	71·8	82·7		

this slope, the crack-extension parameter n can be obtained, according to eqn (9). If this procedure is repeated, the statistical distribution of n can be investigated.

Due to this evaluation procedure, the slope of the linear regression is now

$$\mathcal{K}(\rho) = k_0 + \sum_{\beta=0}^{N-1} c_\beta \log\left(\sum_{j=1}^{M} \frac{1}{M}\rho_{\beta j}\right) \quad (42)$$

in contrast to eqn (19). The coefficients c_β are now described by

$$c_\beta = \frac{6(N-1-2\beta)}{N(N^2-1)} \quad (43)$$

Here again the sum of the coefficients over β is zero.

Because this equation could not be solved in an analogous way to the procedure presented in Section 3, a numerical simulation was performed. From eqn (12) a discrete random distribution of strength values $\sigma_{\beta j}$ for a certain scale parameter σ_β can be calculated by letting $P_{\beta j}$ be a random number between 0 and 1. The index β denotes the number of different loading rates and j the number of tests at one loading rate ($N = 5$ and $M = 10$ for the proposed standardization procedure):

$$\sigma_{\beta j} = \sigma_\beta\left(-\frac{1}{m_*}\ln(1-P_{\beta j})\right) \quad \beta = 0, \ldots, N-1 \quad j = 1, \ldots, M \quad (44)$$

In practice, only the Weibull modulus m of the inert strength is known. Hence, in this calculation the crack-extension parameter n and the Weibull modulus m were varied, and m_* follows from eqn (14). The crack-extension parameter n is running from 10 to 100 and m from 10 to 25. For each combination of n and m one million n values were calculated, which corresponds to one million tests to determine the crack velocity according to the proposed method. The deviation from the true value is henceforth in a rough estimation of the order 10^{-3}.

The distribution function $f(n)$ is shown in Fig. 1 for $m = 10$ and $n = 30$, 50 and 70 (from left to right). This illustrates the increase in the width and decrease in height with increasing n. It is a consequence of the fact that the slope of higher n values is closer to zero, therefore a small variation results in a higher uncertainty and thus a wider distribution as already mentioned.

The result of the Monte-Carlo simulation for the standard deviation is shown in Table 2. If one accepts a standard deviation of about 20% of n as tolerable, one can see that the limiting range of the applicability of the method is $n = 30$ for $m = 10$, $n = 50$ for $m = 15$, $n = 70$ for $m = 20$ and reaches $n = 80$ for $m = 25$. The standard deviation is a function of n_0 and m; the dependence is in a first approximation proportional to $(n_0 + 1)^2/m_*$ (and thus $(n_0 + 1)(n_0 - 2)/m$), which is equal to the result in Section 3.

Figure 2 shows the percentage increase in the standard deviation with increasing n for $m = 10$ (dashed line) and $m = 20$ (solid line). From this diagram the strong dependence of the standard deviation on both n and m is clearly visible.

For n higher than the calculated values given in Table 2 the standard deviations exceed the 20%

Table 2. Standard deviations for a simulation of n-values calculated by using the procedure proposed for the CEN-standard

m/n	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*
10	0·472	2·08	5·07							
15	0·318	1·38	3·24	6·04	10·1					
20	0·240	1·04	2·41	4·42	7·14	10·8	15·6			
25	0·193	0·830	1·92	3·50	5·57	8·28	11·7	15·8		

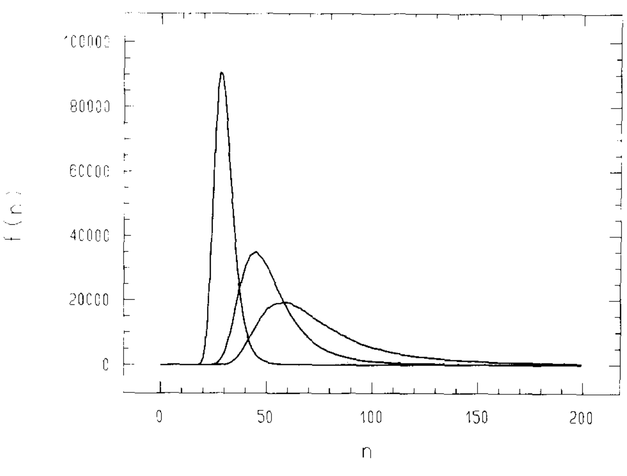

Fig. 1. Distribution function $f(n)$ calculated by the method proposed for the CEN standard for $m = 10$ and $n = 30$, 50, 70 (from left to right). Ordinate: number of n values in an interval with a spacing of one for 10^6 simulated tests.

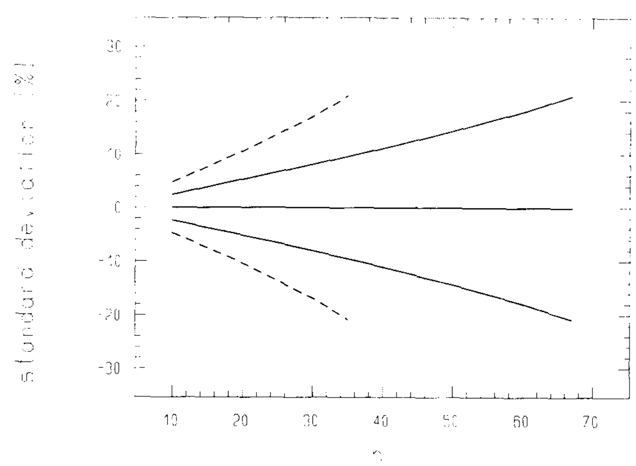

Fig. 2. Relative standard deviation $\Delta n/n$ in percent using the procedure proposed for the CEN-standard. Upper and lower dashed line: for an inert Weibull modulus of $m = 10$, upper and lower solid line: for an inert Weibull modulus of $m = 20$. Central solid line: mean value of n according to eqn (39).

limit and are thus too large to define a useful crack extension parameter value. It should be noted that this is only an effect caused by the Weibull distribution of the bending strength values and not including any experimental difficulties. Even with the best technical equipment, the standard deviation cannot be smaller than the given value.

As a conclusion it is evident that the Weibull modulus m should be known for estimating the limit to which the crack extension parameter n should be determined by four-point-bending.

5 Improved Mathematical Procedures

Instead of choosing the mean value of the measured values at each loading rate, one can choose the median value, $\sigma_{\mathrm{med},\beta}$, by

$$\sigma_{\mathrm{med},\beta} = \tfrac{1}{2}(\sigma_{\beta 5} + \sigma_{\beta 6}) \tag{45}$$

where $\sigma_{\beta 5}$ is the fifth and $\sigma_{\beta 6}$ the sixth measured value after having sorted the ten measured bending strength values at each loading rate in ascending order. This has been discussed by the DIN working group for standardization.[12] Unfortunately, it turns out that this procedure is not an improvement, see Table 3. It was expected that taking the median instead of the mean values is less sensitive to scattering. But the mean values give a better approximation to the true value of n_0 and a smaller standard deviation than the median values.

Another possibility, which was investigated, is to calculate all 10^5 possible regression lines from the ten measured values at five different loading rates. The slopes of these regression fits can be looked at as a set, which can be evaluated by computing its mean value according to eqn (39). A statistical analysis is very computer-time consuming, because 10^5 regression lines have to be calculated just for one simulated experiment. In the numerical calculations, it turned out that using the median value of these 10^5 regression lines instead of the mean value is the better approximation. However, both methods have a higher standard deviation than the evaluation method proposed for the CEN standard.

Table 3. Standard deviations for a simulation of n-values calculated by an evaluation procedure using median values

m/n	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*
10	0·532	2·38	6·00							
15	0·352	1·54	3·65	6·93						
20	0·264	1·15	2·67	4·94	8·08	12·4				
25	0·211	0·912	2·12	3·86	6·23	9·28	13·2	18·3		

5.1 Linear regression of the scale parameters

The main goal is to find the best-suited evaluation method. Although this was not achieved in full generality, a better evaluation procedure than the one proposed for the CEN standard is presented, which needs only slightly more computational effort. The Weibull distribution suggests that one could obtain a promising evaluation procedure by 'correcting' the mean values at each loading rate. Hence, the scale parameter, which is obtained by the maximum-likelihood method, is referred to as

$$\sigma_{\mathrm{sc},\beta} = \left(\sum_{j=1}^{M} \frac{1}{M} (\sigma_{\beta j})^{m_*}\right)^{1/m_*} \tag{46}$$

where M is the number of tests at a loading rate. It can be looked at as a 'Weibull-motivated mean value'. The crack-extension parameter n is obtained by the usual regression fit of the scale parameters.

This method has two advantages: firstly, the standard deviation is smaller than that of the method proposed for the CEN standard. The Monte-Carlo simulation in Table 4 shows the standard deviations for the most interesting case of ten experiments at five different loading rates as proposed for the CEN standard. This procedure improves the range in which one can determine n values. In practice, however, only the inert modulus m is known in advance, and not m_*. Then one should calculate the first approximate n value by inserting m into eqn (46). With the obtained n, m_* follows by eqn (14). With this m_*, a new value for n is obtained by eqn (46). The procedure converges very fast. A simulation with two iterations starting from m shows no difference compared to the results obtained by using m_* from the beginning.

Secondly, the standard deviation can be analytically solved, analogously to the procedure outlined in Section 3. Due to this appealing property, one can calculate the number of specimens, which has to be tested to reach a certain accuracy.

The slope of the linear regression for this evaluation method is given by:

$$\mathcal{K}(\rho) = k_0 + \underbrace{\sum_{\beta=0}^{N-1} c_\beta \log\left(\sum_{j=1}^{M} \frac{1}{M} \rho_{\beta j}^{m_*}\right)^{1/m_*}}_{u(\rho)} \tag{47}$$

Table 4. Standard deviations for a simulation of n-values calculated by the scale parameter of 10 values at each of the 5 loading rates

m/n	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*	*100*
10	0·394	1·73	4·14	7·98						
15	0·262	1·13	2·65	4·90	7·99	12·3				
20	0·196	0·848	1·96	3·58	5·72	8·49	12·0	16·4		
25	0·157	0·676	1·56	2·83	4·50	6·59	9·17	12·3	16·0	20·6

The coefficients c_β are then

$$c_\beta = \frac{6(N-1-2\beta)}{N(N^2-1)} \quad (48)$$

Comparing eqns (47) and (48) with eqns (19) to (25), the procedure to compute the expectation value of k is straightforward:

$$\langle k\rangle = k_0 + \int \mathrm{d}W u(\rho) \quad (49)$$

To calculate the expectation value of k, one has to evaluate the integral:

$$\int \mathrm{d}W u(\rho) = \frac{1}{m_*}\sum_\beta c_\beta \underbrace{\int \mathrm{d}W \log\left(\frac{1}{M}\left(\sum_j \rho_{\beta j}^{m_*}\right)\right)}_{F} \quad (50)$$

Let us now proceed with F, which is identical to F_β for all β:

$$F \equiv F_\beta = \int \prod_{\alpha=0}^{N-1}\prod_{i=1}^{M} \mathrm{d}\rho_{\alpha k} P'(\rho_{\alpha k}) \log \frac{1}{M}(\rho_{\beta 1}^{m^*} + \cdots + \rho_{\beta M}^{m^*}) \quad (51)$$

Because all β are equal, it is possible to set $\beta = 1$ without loss of generality. Then for the integral one obtains:

$$F = \underbrace{\int \prod_{\alpha=1}^{N-1}\prod_{i=1}^{M} \mathrm{d}\rho_{\alpha k} P'(\rho_{\alpha k})}_{1} \times \int \prod_{i=1}^{M} \mathrm{d}\rho_{0i} P'(\rho_{0i}) \log \frac{1}{M}(\rho_{01}^{m*} + \cdots + \rho_{0M}^{m*}) \quad (52)$$

By substituting $x_j = \rho_j^{m_*}$ the integral thus simplifies to

$$F = \int \prod_{i=1}^{M} \mathrm{d}x_i \, e^{-(x_1+\cdots+x_M)} \log \frac{1}{M}(x_1 + \cdots + x_M) \quad (53)$$

Now the number of integrals which have to be solved can be reduced to one by the following coordinate transformation:

$$\begin{aligned} x_1 &= y_1 \\ &\vdots \\ x_{M-1} &= y_{M-1} \\ x_1 + \cdots + x_M &= y_M \end{aligned} \quad (54)$$

With the abbreviation $y = y_M$ thus follows:

$$F = \int_0^\infty \left[\int \mathrm{d}y_1 \cdots \int \mathrm{d}y_{M-1}\right] \mathrm{d}y \, y^{M-1} e^{-y} \log \frac{y}{M} \quad (55)$$

$$y_1 + \cdots + y_{M-1} < 1$$
$$y_i > 0$$

The integrals in the brackets are determined by replacing $\log y/M \to 1$ in eqn (53) and (55), since the relation holds for arbitrary integrands. The integrals in the brackets thus evaluate to $1/\Gamma(M)$.

Hence, the solution arrived at is

$$F = \frac{1}{\Gamma(M)(\ln 10)} \int_0^\infty \mathrm{d}y \, y^{M-1} e^{-y} \ln \frac{y}{M} = \frac{1}{(\ln 10)}\left(\frac{\Gamma'(M)}{\Gamma(M)} - \ln M\right) \quad (56)$$

Corresponding to the evaluation methods outlined in the foregoing sections, and taking into account that the sum over β of c_β is zero, the expectation value of k, $\langle k\rangle$, gives the true value k_0:

$$\langle k\rangle = k_0 + \frac{1}{m_*(\ln 10)}\sum_\beta c_\beta \underbrace{\left(\frac{\Gamma'(M)}{\Gamma(M)} - \ln M\right)}_{\tau} = k_0 \quad (57)$$

The standard deviation can be calculated analogously to eqn (29) by

$$\begin{aligned}(\Delta k)^2 = \langle u(\rho)^2\rangle &= \frac{1}{(m_* \ln 10)^2} \times \int \mathrm{d}W \left(\ln \sum_\alpha \frac{1}{M}\rho_\alpha^{m^*}\right)\left(\ln \sum_\beta \frac{1}{M}\rho_\beta^{m^*}\right) \\ &= \frac{1}{(m_* \ln 10)^2}\sum_\beta c_\beta \sum_\alpha c_\alpha \times \Bigg[\underbrace{\left(\frac{\Gamma'(M)}{\Gamma(M)} - \ln M\right)^2}_{\tau^2} + \delta_{\alpha\beta}\underbrace{\left(\frac{\Gamma''(M)}{\Gamma(M)} - \left(\frac{\Gamma'(M)}{\Gamma(M)}\right)^2\right)}_{\tau_2}\Bigg]\end{aligned} \quad (58)$$

Up to now, one can state that the following two conditions are fulfilled: $\langle k\rangle$ gives the true value k_0 and one can analytically compute the standard deviation. In other words, the evaluation method produces k values, distributed around the true value k_0:

$$\begin{aligned}\langle k\rangle &= k_0 \\ (\Delta k)^2 &= \frac{\tau_2}{m_*^2(\ln 10)^2}\sum_\beta c_\beta^2 = \frac{\tau_2}{m_*^2(\ln 10)^2}\frac{12}{N(N^2-1)}\end{aligned} \quad (59)$$

5.2 Standard deviation of the crack-extension parameter *n*

The calculation is performed analogously to Section 3 and the foregoing section. That is why only the results are presented. The standard deviation of n is obtained by

$$\begin{aligned}(\Delta n)^2 &= \langle (n+1)^2\rangle - \langle n+1\rangle^2 \\ &= (n_0+1)^2(f_2 + 3f_4 - f_2^2 + 5f_6 - 2f_2f_4 + - \cdots)\end{aligned} \quad (60)$$

where the factors f_l up to $l = 6$ are now given by

$$\begin{aligned} f_2 &= \left\langle\left(\frac{u}{k}\right)^2\right\rangle = \left(\frac{n_0+1}{m_* \ln 10}\right)^2 \sum_\beta (c_\beta^2\tau_2) \\ f_4 &= \left\langle\left(\frac{u}{k}\right)^4\right\rangle = \left(\frac{n_0+1}{m_* \ln 10}\right)^4 \times \sum_\beta (c_\beta^4(\tau_4 - 3\tau^2 + 4\tau\tau_3 + 6\tau^2\tau_2) + 3c_\beta^2\tau_2^2) \\ f_6 &= \left\langle\left(\frac{u}{k}\right)^6\right\rangle = \left(\frac{n_0+1}{m_* \ln 10}\right)^6 \times \sum_\beta (a(c_\beta^6) + 15b(c_\beta^4 c_\beta^2) + 20f(c_\beta^3)^2 + 90g(c_\beta^2)^3)\end{aligned} \quad (61)$$

with the coefficients a to g:

$$a = 15\tau^4\tau_2 + 20\tau^3\tau_3 - 180\tau^2\tau_2^2 + 15\tau^2\tau_4 - 120\tau\tau_2\tau_3 + 30\tau_2^3 - 15\tau_2\tau_4 - 10\tau_3^2 + 6\tau\tau_5 + \tau_6$$
$$b = \tau_2(6\tau^2\tau_2 + 4\tau\tau_3 - 3\tau_2^2 + \tau_4)$$
$$f = \tfrac{1}{2}(9\tau^2\tau_2^2 + 6\tau\tau_2\tau_3 + \tau_3^2)$$
$$g = \frac{\tau_2^3}{6} \quad (62)$$

The factors τ_l are given by the following equation:

$$\tau_l = \sum_{i=0}^{l} \binom{l}{i} (-1)^i \left[\frac{\Gamma^{(i)}(M)}{\Gamma(M)} - \left(\frac{\Gamma'(M)}{\Gamma(M)}\right)^i\right] (\ln M)^{1-i} \quad (63)$$

where $\Gamma^{(i)}$ denotes the ith derivative of the gamma function.

The main result of this section is that it is possible to calculate the number of loading rates N and the number of tests M at each loading rate to obtain a given accuracy. If one is interested only in a rough estimation, one should just use the first approximation,

$$\Delta n = \frac{(n_0+1)^2}{m_*(\ln 10)} \left(\sum_\beta (c_\beta^2 \tau_2)\right)^{1/2} = \frac{(n_0+1)^2(\tau_2)^{1/2}}{m_*(\ln 10)} \left(\frac{12}{N(N^2-1)}\right)^{1/2} \quad (64)$$

which provides an accuracy

$$\frac{\Delta n(\text{approx.}) - \Delta n(\text{Monte-Carlo})}{\Delta n(\text{Monte-Carlo})}$$

of 17% in the worst case for a standard deviation Δn smaller than 20% of n ($\Delta n/n < 0{\cdot}2$).

6 Different Experimental Procedures

The most effective way to essentially increase the accuracy is to test more specimens. If the number of loading rates is increased to six, one needs ten additional tests, but they last ten times longer. On the other hand, if the number of tests at a certain loading rate is increased to 15, 25 additional bending strength tests have to be performed. These additional tests are less time consuming, but need more preparation work for the higher number of specimens.

In fact, both variants are nearly equal. Testing at more loading rates is only slightly better and reduces the standard deviation in comparison to testing at five loading rates by a factor of about 0·75, testing 15 instead of 10 specimens at one loading rate by a factor of about 0·8, both evaluated using the method of the scale parameters.

The authors want to point out that the knowledge of the Weibull modulus is a necessary condition to estimate the reliability of determining n values. When a new material is produced, the Weibull modulus of the ceramic of course is not known in advance. Then it is recommended that the Weibull modulus m of the inert strength is obtained by performing 30 tests at a loading rate, high enough to provide from crack extension. This was shown to be a reasonable number of tests to get a sufficiently exact value for the Weibull modulus.[13] With this Weibull modulus, the number of tests and loading rates respectively can be calculated, which is necessary to obtain a certain accuracy. Then these tests have to be performed at loading rates chosen in the way that the mean bending strength values show a decrease and keep off the plateau region of the inert strength values.

7 Conclusion

There exists a certain upper bound for the n value, dependent the Weibull modulus m, below which reliable results for n can be obtained by four-point-bending tests. It is possible to shift this upper bound to higher values and thus to enlarge the range of reliable results by the new evaluation procedure proposed. This evaluation needs only slightly more computational effort.

If the n values are outside the limited range of applicability, i.e. they are higher than the upper bound, with the new evaluation procedure an increased number of specimens has to be tested. The number of specimens necessary, to obtain a certain accuracy can be computed according to eqn (60) to (63). High n values in combination with a low Weibull modulus m are questionable and only reliable if an extensive amount of specimens is tested. Because of the high standard deviation the lifetime calculations can lead to extremely different results for this case.

If the Weibull modulus is not known, it is recommended that 30 tests are performed at a very high loading rate to determine m of the inert strength. With the knowledge of m, one knows the theoretically lowest margin of error of n for a certain number of tests at a certain number of loading rates.

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